JOURNAL OF COMBINATORIAL THEORY, Series B 41, 306–331 (1986)

On the Chromatic Index of Multigraphs and a Conjecture of Seymour (I)

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Communicated by the Managing Editors

Received November 9, 1984

Let $\mathscr{G} = (V, E, w)$ be a multigraph, where V is a set of vertices, E is a set of edges, and w is a vector of edge multiplicities. It is well known that ρ, the maximum degree of $\mathscr{G}$, is a lower bound on the cardinality of a proper edge coloring of $\mathscr{G}$. Another lower bound is given by $\kappa = \max\{w(E(S))/((|S|-1)/2) \mid S \subseteq V,\ |S|$ odd and $|S| \neq 1\}$, where $w(E(S))$ is the number of edges both ends of which belong to S. P. D. Seymour [*Proc. London Math. Soc. (3)* **38** (1979), 423–460] has made the conjecture that the minimum number of colors in a proper edge coloring of $\mathscr{G}$ is less than or equal to $\max\{\rho+1, \lceil\kappa\rceil\}$, where $\lceil\kappa\rceil$ denotes the least integer greater than or equal to κ. In this paper we show that Seymour's conjecture can be reduced to a conjecture about critical nonseparable graphs (in the sense of matching theory). We also show that the latter conjecture is verified in the case of outerplanar graphs, thus proving that Seymour's conjecture holds for outerplanar graphs.

0. INTRODUCTION

Let $\mathscr{G} = (V, E, w)$ be a *multigraph*, where V is the vertex set of $\mathscr{G}$, E is the edge set of $\mathscr{G}$, and w is a vector of multiplicities (that is, w_e is the multiplicity of edge e). We shall sometimes use the term *graph* instead of multigraph. A *proper coloring* of the edges of $\mathscr{G}$ is an assignment of colors to the edges with the property that no two adjacent edges have the same color. The *chromatic index* of $\mathscr{G}$ is the minimum number of colors in a proper edge coloring of $\mathscr{G}$. It is clear that the chromatic index of $\mathscr{G}$ is greater than or equal to the maximum degree of a vertex in $\mathscr{G}$ (which we shall denote by ρ). There is, however, another lower bound on the chromatic index of $\mathscr{G}$. Let $E(S) = \{e \mid$ both ends of e belong to $S\}$, and

* The work of the author was partially supported by NSF Grant 81-13534 to Cornell University and by Grant A2474 from NSERCC.

0095-8956/86 $3.00

$w(E(S)) = \sum \{w_e \mid e \in E(S)\}$. The second lower bound is given by $\lceil \kappa \rceil$, where

$$\kappa = \max \left\{ \frac{w(E(S))}{(|S|-1)/2} \;\middle|\; S \subseteq V,\ |S| \text{ odd and } |S| \neq 1 \right\},$$

and $\lceil \kappa \rceil$ denotes the least integer greater than or equal to κ. $\lceil \kappa \rceil$ is a lower bound on the chromatic index of $\mathscr{G}$ because if S is any subset of V of odd cardinality, no more than $\frac{1}{2}(|S|-1)$ edges may have the same color in the subgraph of $\mathscr{G}$ induced by S.

Seymour [7] has made the conjecture that the chromatic index of $\mathscr{G}$ is less than or equal to $\max\{\rho+1, \lceil \kappa \rceil\}$. For simple graphs, that is, for graphs such that $w_e = 1$ for every $e \in E$, this conjecture reduces to Vizing's theorem (see [7]). Seymour's conjecture is actually a statement about the difference between the optimal value of an integer program and that of its linear programming relaxation. In order to show this, we need to define a matching. A *matching* $\mathscr{M}$ of $\mathscr{G}$ is a subset of $\{e \in E \mid w_e > 0\}$ such that no two edges in $\mathscr{M}$ are adjacent. It is easily seen that in a proper edge coloring of $\mathscr{G}$, the edges which have been assigned a given color constitute a matching. The edge coloring problem can thus be formulated as a covering problem

$$\begin{aligned} &\min\ 1.y \\ &\qquad \text{such that} \quad yM \geqslant w \\ &\qquad\qquad\qquad y \geqslant 0,\ y \text{ integral}, \end{aligned} \tag{IP}$$

where the rows of $\mathscr{M}$ are the incidence vectors of the matchings of $\mathscr{G}$. It follows from Edmonds' matching theorem that the optimal value of the linear relaxation of (IP) is equal to $\max\{\rho, \kappa\}$ (see also Theorem 3.6 in [7]). The number $\max\{\rho, \kappa\}$ is sometimes called the *fractional chromatic index* of $\mathscr{G}$. It follows from these remarks that if Seymour's conjecture were true, the difference between the optimal value of (IP) and that of its linear programming relaxation would be less than or equal to 1.

The present paper can thus be viewed as a contribution to the study of integer programming problems whose optimal value is close to that of their linear programming relaxation. Baum and Trotter [1] have studied problems with the integer rounding property, and the relationship between the latter property and the integral decomposition property has provided the rationale for the conjecture and results presented in our paper. On the other hand, conjectures similar to the one which we are studying have been published at least since 1973 (see Goldberg [4]), and in particular, Conjecture 1 in Goldberg [5] is almost identical to Seymour's conjecture. In Goldberg [5] it is also shown that if the chromatic index of $\mathscr{G}$ is greater than $\frac{1}{8}(9\rho+6)$, then the chromatic index is equal to $\lceil\kappa\rceil$. This result lends weight to Seymour's conjecture, since it implies that the chromatic index of $\mathscr{G}$ is given by $\lceil\kappa\rceil=\max\{\rho+1,\lceil\kappa\rceil\}$ whenever $\kappa>\frac{1}{8}(9\rho+6)$.

The above formulation of the edge coloring problem as an integer programming problem suggests that Seymour's conjecture can be reduced to a conjecture about critical graphs. Critical graphs play an important role in matching theory (see Pulleyblank [6]). Let $\mathscr{G}$ be a graph with an odd number of vertices ($|V|>1$), and v be any vertex of $\mathscr{G}$. A *near perfect matching* deficient at v is a matching $\mathscr{M}$ of $\mathscr{G}$ such that every node of $\mathscr{G}$ but v is adjacent to an edge of $\mathscr{M}$. $\mathscr{G}$ is said to be *critical* if for every vertex v of $\mathscr{G}$, there is a near perfect matching of $\mathscr{G}$ deficient at v. The purpose of this paper is twofold: first to show that Seymour's conjecture can be reduced to a conjecture about critical graphs; second, to show that this new conjecture is actually satisfied in the case where $\mathscr{G}$ is an outerplanar multigraph. In Section 1 we prove the reduction. Sections 2 and 3 are devoted to proving that Seymour's conjecture is satisfied for outerplanar multigraphs.

Throughout the paper we shall denote $w(E(S))/((|S|-1)/2)$ by κ_S, where S is a subset of odd cardinality with $|S|>1$. We shall denote by $\mathscr{G}_S$ the subgraph induced by the set S of vertices, and by $\delta(S)$ the coboundary of S. If S consists of a single vertex, v, we use the notation $\delta(v)$ instead of $\delta(\{v\})$. Finally, let y be any vector in $\mathbb{R}^E$ and F be any subset of E. We shall denote $\sum\{y_e \mid e\in F\}$ by $y(F)$.

1. Critical Graphs and Seymour's Conjecture

In order to show the relationship between critical graphs and the edge coloring problem, we shall recall the description of the matching polytope given by Edmonds [3] and Pulleyblank [6]. Let $\mathscr{G}$ be a multigraph. The matching polytope (which we denote by $\mathscr{P}$) is the convex hull of the incidence vectors of all matchings of $\mathscr{G}$. The following theorem may be found in Pulleyblank [6].

THEOREM 1.1. *$\mathscr{P}$ is the set of solutions of the following system of inequalities*:

(1) $x(\delta(v))\leqslant 1$ *for each* $v\in V$;

(2) $x(E(S))\leqslant(|S|-1)/2$ *for each subset S of V such that $|S|$ is odd and greater than* 1;

(3) $x_e\geqslant 0$ *for every* $e\in E$.

Furthermore, an inequality of type 2 is a facet of the matching polytope if and only if the subgraph $\mathscr{G}_S$ is critical and nonseparable.

Let $\mathscr{G}_S = (S, E, w)$ be a multigraph whose vertex set has odd cardinality, and let ρ denote the maximum degree of $\mathscr{G}_S$. We say that $\mathscr{G}_S$ is a κ-graph if κ_S is greater than ρ and than κ_T for every proper subset T of S such that $|T|$ is odd and $|T| \geqslant 3$. In what follows, we shall also need the definition of augmented κ-graph. For any κ-graph $\mathscr{G}_S$, we let ρ^a be the largest integer less than κ_S. We form an *augmented κ-graph* as follows: add a distinguished vertex v^a to the vertex set of $\mathscr{G}_S$, and for every vertex v of S, join v and v^a by an edge of multiplicity $\rho^a - d(v)$, where $d(v)$ is the degree of vertex v in $\mathscr{G}_S$. Thus in an augmented κ-graph all vertices belonging to S have the same degree. We let $\mathscr{G}_S^a$ denote the augmented κ-graph corresponding to $\mathscr{G}_S$.

THEOREM 1.2. *Let $\mathscr{G}_S = (S, E, w)$ be a κ-graph. Then $\mathscr{G}_S$ is critical and nonseparable.*

Proof. Let us assume that the theorem fails. Then the inequality

$$x(E(S)) \leqslant \frac{|S|-1}{2}$$

is not a facet of the matching polytope. By Theorem 1.1 and Farkas' lemma, we may conclude that there exist nonnegative real numbers λ_v (for $v \in S$) and λ_T (for T a proper subset of S with $|T|$ odd and greater than 1) such that

$$\frac{\alpha_S}{(|S|-1)/2} \leqslant \sum \lambda_v \alpha_v + \sum \lambda_T \frac{\alpha_T}{(|T|-1)/2} \tag{1.3}$$

and

$$\sum \lambda_v + \sum \lambda_T = 1.$$

Here α_v and α_T denote the incidence vectors of $\delta(v)$ and $E(T)$, respectively. By taking the inner product of w with inequality 1.3, we obtain

$$\frac{w(E(S))}{(|S|-1)/2} \leqslant \sum \lambda_v w(\delta(v)) + \sum \lambda_T \frac{w(E(T))}{(|T|-1)/2},$$

and therefore

$$\kappa_S \leqslant \sum \lambda_v \rho + \sum \lambda_T \kappa_T < \left(\sum \lambda_v + \sum \lambda_T\right) \kappa_S = \kappa_S$$

since $\mathscr{G}_S$ is a κ-graph.

We have thus reached a contradiction, and we conclude that $\mathscr{G}_S$ is critical and nonseparable. ∎

The previous theorem implies in particular that for every vertex v of S, there exists a near perfect matching of $\mathscr{G}_S$ which is deficient at v. It is tempting to make the conjecture that the edge set of $\mathscr{G}_S$ can be partitioned into $\lceil \kappa_S \rceil$ matchings, all of which (with the possible exception of one) are near perfect matchings of $\mathscr{G}_S$. We shall see in Section 3, however, that this conjecture is false; the graph of Fig. 3.2, among others, is a counterexample to this conjecture. Although this simple conjecture does not hold, it seems desirable to replace Seymour's conjecture by a conjecture about κ-graphs or augmented κ-graphs, since the latter occur as "subgraphs" of $\mathscr{G}$ whenever $\kappa > \rho$ for a given multigraph $\mathscr{G}$. In the discussion following Lemma 1.4, we shall argue that κ-graphs or augmented κ-graphs may also arise in the coloring of edges of multigraphs with the property that $\rho \geqslant \kappa$.

LEMMA 1.4. *Let $\mathscr{G} = (V, E, w)$ be a multigraph such that $\kappa \leqslant \rho$, where ρ is the maximum degree of $\mathscr{G}$ and*

$$\kappa = \max \left\{ \frac{w(E(T))}{(|T|-1)/2} \;\middle|\; T \subseteq V,\ |T| \text{ odd and } |T| \geqslant 3 \right\}.$$

Then there exists a matching $\mathscr{M}$ of $\mathscr{G}$ such that each vertex of degree ρ is incident upon an edge belonging to $\mathscr{M}$.

Proof. Let $w' = w/\rho$. Then w' satisfies the following equalities and inequalities:

$$w'(\delta(v)) = 1 \qquad \text{for every } v \text{ such that } d(v) = \rho;$$

$$w'(\delta(v)) \leqslant 1 \qquad \text{for every } v \text{ such that } d(v) < \rho;$$

$$w'(E(T)) \leqslant \frac{|T|-1}{2} \qquad \text{for every } T \subseteq V \text{ with } |T| \text{ odd and } |T| \geqslant 3;$$

and

$$w'_e \geqslant 0 \qquad \text{for} \quad e \in E.$$

This implies that the set of solutions of the following system is not empty:

$$\begin{aligned} &x(\delta(v)) = 1 && \text{for every } v \text{ such that } d(v) = \rho;\\ &x(\delta(v)) \leqslant 1 && \text{for every } v \text{ such that } d(v) < \rho;\\ &x(E(T)) \leqslant \frac{|T|-1}{2} && \text{for every } T \subseteq V \text{ with } |T| \text{ odd and } |T| \geqslant 3; \end{aligned} \qquad \text{(S)}$$

and

$$x_e \geqslant 0 \qquad \text{for} \quad e \in E.$$

But the set of solutions of system (S) is a face of P, the matching polyhedron; thus the set of solutions of (S) contains a (0, 1) vector (say x^0) which is the incidence vector of a matching of $\mathscr{G}$. Let $\mathscr{M}$ be the matching whose incidence vector is x^0. Since $x^0(\delta(v))=1$ for every v such that $d(v)=\rho$, every vertex of degree ρ is incident upon an edge of $\mathscr{M}$. ∎

The preceding lemma has the following interpretation: given a multigraph $\mathscr{G}$ verifying the hypotheses of the lemma, it is possible to remove from the edge set of $\mathscr{G}$ a matching (actually, the matching mentioned in the conclusion of the lemma) such that the resulting multigraph $\mathscr{G}'$ has maximum vertex degree equal to $\rho-1$. Let ρ' and κ' denote respectively the maximum vertex degree and maximum "odd set quotient" of $\mathscr{G}'$. Then either $\rho'=\rho-1\geqslant\kappa'$, in which case Lemma 1.4 can be applied once more, or $\rho'<\kappa'$, in which case $\mathscr{G}'$ contains a κ-graph. By applying Lemma 1.4 as many times as possible, either one of two situations may arise: the edge set of $\mathscr{G}$ can be partitioned into ρ matchings, or the edge set of $\mathscr{G}$ can be partitioned into $\rho-\rho^1$ matchings and a subgraph $\mathscr{G}^1$ of maximum degree ρ^1 such that $\rho^1<\kappa^1$.

So let us consider a multigraph $\mathscr{G}=(V, E, w)$ with $\kappa>\rho$. It is possible to decompose $\mathscr{G}$ by a "shrinking" operation analogous to the one used in the matching algorithm. We shall first define what we mean by "shrinking."

Definition 1.5. Let $\mathscr{G}=(V, E, w)$ be a multigraph, and let S be a proper subset of V. Let us define a graph $\mathscr{G}^1=(V^1, E^1, w^1)$ as follows:

$$V^1=(V\setminus S)\cup\{v_P\},\qquad \text{where } v_P \text{ is a new vertex;}$$

$$E^1=E(V\setminus S)\cup\{\{v, v_P\}\mid \exists\{v, v'\}\in E \text{ such that } v\in V\setminus S \text{ and } v'\in S\};$$

and

$$w_e^1=\begin{cases} w_e & \text{for } e\in E(V\setminus S)\\ w(\delta(v, S)) & \text{for } e=\{v, v_P\}, \text{ where } \delta(v, S)=\{\{v, v'\}\mid v'\in S\}.\end{cases}$$

We say that $\mathscr{G}^1$ is the graph obtained from $\mathscr{G}$ by *shrinking* the set S.

Let us now return to the case $\kappa>\rho$. Then it is possible to find a subset S of V such that $\kappa_S=\kappa$ and $\kappa_S>\kappa_T$ for every T with $T\subset S$, $|T|$ odd and $|T|\geqslant 3$. If $S=V$, $\mathscr{G}$ is a κ-graph. If $|S|=|V|-1$, let $\{v^a\}=V\setminus S$. Then $\mathscr{G}$ can be considered as a subgraph of $\mathscr{G}_S^a$. Finally, if $|S|\leqslant|V|-2$, let $\mathscr{G}^1$ be the graph obtained from $\mathscr{G}$ by shrinking S. Since $|S|\geqslant 3$, the cardinality of V^1 is smaller than that of V. On the other hand, we define $\mathscr{G}^2=(V^2, E^2, w^2)$ to be the augmented κ-graph $\mathscr{G}_S^a$. The cardinality of V^2 is less than that of V, since $|S|\leqslant|V|-2$. Let $\delta^1(v_P)$ and $\delta^2(v^a)$ be respectively the stars of v_P in $\mathscr{G}^1$ and of v^a in $\mathscr{G}^2$. The following lemma relates the chromatic index of $\mathscr{G}$ to the chromatic indices of $\mathscr{G}^1$ and $\mathscr{G}^2$.

LEMMA 1.6. *The chromatic index of $\mathscr{G}$ is less than or equal to* $\max\{l, k\}$, *where l and k are the chromatic indices of $\mathscr{G}^1$ and $\mathscr{G}^2$, respectively.*

Proof. We let v_P be the vertex of $\mathscr{G}^1$ which "replaces" S. In a similar fashion, let $\mathscr{G}^3$ be the graph obtained from $\mathscr{G}$ by shriking $V\backslash S$, and let v^a denote the vertex which "replaces" $V\backslash S$ in $\mathscr{G}^3$. It is clear that $\mathscr{G}^2$ and $\mathscr{G}^3$ have the same set of vertices and the same set of edges, and that $w^3 \leqslant w^2$ (i.e., $w_e^3 \leqslant w_e^2$ for every edge e). Therefore the chromatic index of $\mathscr{G}^3$ is less than or equal to k. Since the chromatic index of $\mathscr{G}^1$ is l, there exist matchings $\mathscr{M}_1^1,..., \mathscr{M}_l^1$ of $\mathscr{G}^1$ such that $\sum_{i=1}^{l} x^i = w^1$, where x^i is the incidence vector of $\mathscr{M}_i^1$. Let $r = w(\delta^1(v_P)) = w(\delta^3(v^a)) = w(\delta(s))$, where $\delta^1(v_P)$ and $\delta^3(v^a)$ denote respectively the star of v_P in $\mathscr{G}^1$ and the star of v^a in $\mathscr{G}^3$. Without loss of generality, we may assume that each of the matchings $\mathscr{M}_1^1,..., \mathscr{M}_r^1$ contains an edge of $\delta^1(v_P)$. Thus $|\mathscr{M}_i^1 \cap \delta^1(v_P)| = 1$ for $i = 1,..., r$ and $|\mathscr{M}_i^1 \cap \delta^1(v_P)| = 0$ for $i > r$. In a similar fashion, there exist matchings $\mathscr{M}_1^3,..., \mathscr{M}_k^3$ such that

$$\sum_{j=1}^{k} y^j = w^3, \qquad \text{where } y^j \text{ is the incidence vector of } \mathscr{M}_j^3;$$

$$|\mathscr{M}_j^3 \cap \delta^3(v^a)| = 1 \qquad \text{for} \quad j = 1,..., r;$$

$$|\mathscr{M}_j^3 \cap \delta^3(v^a)| = 0 \qquad \text{for} \quad j > r.$$

Again without loss of generality, we may assume that for $i = 1,..., r$, there exists an edge $e_i = \{u_i, v_i\}$ in $\mathscr{G}$ such that $\mathscr{M}_i^1$ contains $\{u_i, v_P\}$ and $\mathscr{M}_i^3$ contains $\{v^a, v_i\}$. If $l \geqslant k$, we define a collection of matchings of $\mathscr{G}$ as follows:

$$\mathscr{M}_i = (\mathscr{M}_i^1 \backslash \{u_i, v_p\}) \cup (\mathscr{M}_i^3 \backslash \{v^a, v_i\}) \cup \{e_i\} \qquad \text{for} \quad i = 1,..., r;$$

$$\mathscr{M}_i = \mathscr{M}_i^1 \cup \mathscr{M}_i^3 \qquad \text{for} \quad i = r+1,..., k,$$

and

$$\mathscr{M}_i = \mathscr{M}_i^1 \qquad \text{for} \quad i > k.$$

The collection of matchings is defined similarly if $k > l$. It is straightforward to verify that $\sum_{i=1}^{\max\{l,k\}} z^i = w$, where z^i is the incidence vector of $\mathscr{M}_i$. Therefore $\{\mathscr{M}_i \mid i = 1, 2,..., \max\{l, k\}\}$ is a proper edge coloring of $\mathscr{G}$, and this completes the proof of the lemma. ∎

The following lemma relates the fractional chromatic index of $\mathscr{G}$ (that is, $\max\{\rho, \kappa\}$) to those of $\mathscr{G}^1$ and $\mathscr{G}^2$.

LEMMA 1.7. *Let ρ, ρ^1, and ρ^2 be the maximum degrees of $\mathscr{G}$, $\mathscr{G}^1$, and $\mathscr{G}^2$, respectively. Similarly let*

$$\kappa = \max\{\kappa_T \mid T \subseteq V,\ |T| \text{ odd and } |T| \geqslant 3\},$$

$$\kappa^1 = \max\{\kappa^1_T \mid T \subseteq V^1,\ |T| \text{ odd and } |T| \geqslant 3\}$$

and

$$\kappa^2 = \max\{\kappa^2_T \mid T \subseteq V^2,\ |T| \text{ odd and } |T| \geqslant 3\}.$$

Then we have $\rho^1 \leqslant \rho$, $\rho^2 < \kappa$, $\kappa^1 \leqslant \kappa$, and $\kappa^2 = \kappa$ (i.e., the fractional chromatic index of $\mathscr{G}$ is equal to that of $\mathscr{G}^2$ and greater than or equal to that of $\mathscr{G}^1$).

Proof. It is clear that the degree in $\mathscr{G}^1$ of a vertex $v \in V \setminus S$ is less than or equal to ρ. On the other hand, $d^1(v_P)$, the degree of v_P in $\mathscr{G}^1$, is equal to $w(\delta(S))$. We thus have

$$d^1(v_P) = w(\delta(S)) \leqslant \rho\ |S| - 2w(E(S)) < \rho\ |S| - \rho(|S| - 1) = \rho.$$

Therefore ρ^1, the maximum degree of a vertex in $\mathscr{G}^1$, is less than or equal to ρ. By definition of an augmented κ-graph, the degree in $\mathscr{G}^2$ of any vertex belonging to S is equal to ρ^a, which is smaller than κ. We also have

$$\begin{aligned} d^2(v^a) &= \sum_{v \in S} (\rho^a - d(v)) \qquad \text{(where } d(v) \text{ is the degree of } v \text{ in } \mathscr{G}_S) \\ &= \rho^a\ |S| - 2w(E(S)) \\ &= \rho^a + (\rho^a - \kappa_S)(|S| - 1) \\ &< \rho^a. \end{aligned}$$

We conclude that ρ^2, the maximal degree of a vertex in $\mathscr{G}^2$, is strictly less than κ.

Let T be a subset of V^1 such that $|T|$ is odd and $|T| > 1$. If $v_P \notin T$,

$$\kappa^1_T = \frac{w^1(E(T))}{(|T| - 1)/2} = \kappa_T \leqslant \kappa$$

by definition of κ. If $v_P \in T$, let $W = (T \setminus \{v_P\}) \cup S$. Then

$$\begin{aligned} \kappa^1_T &= \frac{w(E(W)) - w(E(S))}{(|W| - |S|)/2} \\ &= \frac{\kappa_W((|W| - 1)/2) - \kappa_S((|S| - 1)/2)}{(|W| - |S|)/2} \\ &\leqslant \frac{\kappa_S((|W| - 1)/2 - (|S| - 1)/2)}{(|W| - |S|)/2} \\ &= \kappa_S = \kappa. \end{aligned}$$

Thus $\kappa^1 = \max\{\kappa_T^1 \mid T \subseteq V^1,\ |T| \text{ odd and } |T| > 1\}$ is less than or equal to κ.

Finally we show that $\kappa^2 \leqslant \kappa$. Let T be a proper subset of V^2 of odd cardinality such that $|T| \geqslant 3$. If T is a subset of S, we have $\kappa_T^2 = \kappa_T \leqslant \kappa$. On the other hand, if v^a belongs to T and $|S \setminus T| \geqslant 3$, we have

$$\begin{aligned}
2w^2(E(T)) &= \sum_{v \in T} d^2(v) - w(\delta(T)) \\
&= \rho^a(|T|-1) + d^2(v^a) - \left\{ \sum_{v \in S \setminus T} d^2(v) - 2w(E(S \setminus T)) \right\} \\
&= \rho^a(|T|-1) + \{\rho^a\,|S| - \kappa_S(|S|-1)\} \\
&\quad - \{\rho^a(|S|-|T|+1) - \kappa_{S \setminus T}(|S|-|T|)\} \\
&= 2\rho^a(|T|-1) - \kappa_S(|S|-1) + \kappa_{S \setminus T}(|S|-|T|) \\
&\leqslant 2\rho^a(|T|-1) - \kappa_S(|S|-1) + \kappa_S(|S|-|T|) \\
&= (2\rho^a - \kappa_S)(|T|-1).
\end{aligned}$$

Therefore, $\kappa_T^2 \leqslant 2\rho^a - \kappa_S < \rho^a$, since $\kappa_S > \rho^a$. If $v^a \in T$ and $|S \setminus T| = 1$, we can show in a similar fashion that $\kappa_T^2 < \rho^a$. Since $\kappa_S^2 = \kappa_S = \kappa$ and $\kappa_T^2 < \kappa$ for every subset T of V^2 with $T \neq S$, $|T|$ odd and $|T| \geqslant 3$, we conclude that $\kappa^2 = \kappa$; that is, the fractional chromatic index of $\mathscr{G}^2$ is equal to the fractional chromatic index of $\mathscr{G}$. ∎

From Lemmas 1.6 and 1.7, it is clear that augmented κ-graphs play an important role in the edge coloring problem. In view of Lemma 1.4, it is reasonable to think that Seymour's conjecture holds for all graphs if and only if it holds for augmented κ-graphs. Theorem 1.8 states that this is indeed the case.

THEOREM 1.8. *Let us assume that for any augmented κ-graph $\mathscr{G}$, the chromatic index of $\mathscr{G}$ is given by $\lceil \kappa \rceil$. Then Seymour's conjecture holds for all multigraphs.*

Proof. Let $\mathscr{G} = (V, E, w)$ be an arbitrary multigraph, and let $k = \max\{\rho + 1, \lceil \kappa \rceil\}$. The proof is by induction on k and on $|V|$.

For $k = 1$ or $|V| \leqslant 2$, the theorem is trivially true. Therefore we assume that the theorem holds for all multigraphs $\mathscr{G}'$ such that $\max\{\rho' + 1, \lceil \kappa' \rceil\} < k$. Among all multigraphs $\mathscr{G}'$ such that $\max\{\rho' + 1, \lceil \kappa' \rceil\} = k$, we assume that the theorem holds for all multigraphs with $|V'| < |V|$. There are several cases to consider:

(a) $\kappa > \rho$

Let S be a subset of V for which $\kappa_S = \kappa$, $\kappa_S > \kappa_T$ for every $T \subset S$, $|T|$ odd and $|T| \geqslant 3$. Again there are two subcases to consider:

(i) $|S| = |V|$ *or* $|V| - 1$. In this case we can embed $\mathcal{G}$ into the augmented κ-graph $\mathcal{G}_S^a = (S \cup \{v^a\}, E^a, w^a)$. The hypothesis of the theorem enables us to conclude that the chromatic index of $\mathcal{G}$ is $\lceil \kappa \rceil = \max\{\rho + 1, \lceil \kappa \rceil\}$.

(ii) $|S| \leqslant |V| - 2$. In this case we can decompose $\mathcal{G}$ into two multigraphs $\mathcal{G}^1$ and $\mathcal{G}^2$ such that $|V^i| < |V|$ for $i = 1, 2$ (see the discussion which follows Definition 1.5). By Lemma 1.7, $\max\{\rho^1 + 1, \lceil \kappa^1 \rceil\} \leqslant \lceil \kappa \rceil$ and $\max\{\rho^2 + 1, \lceil \kappa^2 \rceil\} = \lceil \kappa \rceil$. Hence by the induction hypothesis the chromatic index of $\mathcal{G}^2$ is $\lceil \kappa \rceil$ and the chromatic index of $\mathcal{G}^1$ is less than or equal to $\lceil \kappa \rceil$. By Lemma 1.6, the chromatic index of $\mathcal{G}$ is given by $\lceil \kappa \rceil = \max\{\rho + 1, \lceil \kappa \rceil\}$.

(b) $\kappa \leqslant \rho$

By Lemma 1.4, there exists a matching $\mathcal{M}$ such that every vertex of degree ρ is incident upon an edge belonging to $\mathcal{M}$. Let $\mathcal{G}' = (V, E, w - x)$, where x is the incidence vector of $\mathcal{M}$. Clearly $\rho' = \rho - 1$ and $\kappa'_T \leqslant \rho$ for every odd subset T of V. Therefore

$$\max\{\rho' + 1, \lceil \kappa' \rceil\} = \rho' + 1 < \rho + 1 = \max\{\rho + 1, \lceil \kappa \rceil\}.$$

By the induction hypothesis the chromatic index of $\mathcal{G}'$ is less than or equal to $\rho' + 1$. We conclude that the chromatic index of $\mathcal{G}$ is less than or equal to $\rho + 1 = \max\{\rho + 1, \lceil \kappa \rceil\}$. This completes the proof of the theorem. ∎

In effect, Theorem 1.8 states that Seymour's conjecture can be replaced by the following one:

Conjecture 1.9. Let $\mathcal{G} = (V, E, w)$ be an augmented κ-graph. Then the chromatic index of $\mathcal{G}$ is equal to $\lceil \kappa \rceil$.

Although the class of augmented κ-graphs is a relatively small class of multigraphs, it would be desirable to reformulate Conjecture 1.9 in terms of κ-graphs since the latter are shrinkable while augmented κ-graphs are not. The structure of shrinkable graphs is well known (see Pulleyblank [6]), and is much simpler than that of arbitrary graphs. First of all, we observe that when an augmented κ-graph $\mathcal{G}_S^a$ is such that $\kappa_S \leqslant \rho + 1$ and $d(v) = \rho$ for every $v \in S$, the edge set of $\mathcal{G}_S^a$ is identical to the edge set of $\mathcal{G}_S$. Therefore in this case Conjecture 1.9 reduces to a conjecture about κ-graphs. On the other hand, when the degree of v^a in $\mathcal{G}_S^a$ is greater than zero, we would like to remove from $\mathcal{G}_S^a$ a perfect matching such that the resulting multigraph, $\mathcal{G}'$, satisfies $\kappa' \leqslant \lceil \kappa \rceil - 1$. The following lemma shows that in order to choose the required mathing, it suffices to check, for every W contained in S, the number of edges in the matching both ends of which belong to W.

LEMMA 1.10. *Let* $\mathcal{G}_S^a = (S \cup \{v^a\}, E^a, w)$ *be an augmented* κ*-graph, and* *let us assume that there exists a perfect matching $\mathcal{M}$ (whose incidence vector we denote by x) such that*

$$\kappa'_W = \frac{(w-x)(E(W))}{(|W|-1)/2}$$

is less than or equal to ρ^a for every $W \subseteq S$, $|W|$ odd and $|W| \geqslant 3$. Then the fractional chromatic index of the multigraph $\mathcal{G}' = (S \cup \{v^a\}, E^a, w-x)$ is less than or equal to ρ^a.

Proof. Let ρ' denote the maximum degree of a vertex in $\mathcal{G}'$. Clearly $\rho' = \rho^a - 1$. Since by hypothesis $\kappa'_T \leqslant \rho^a$ for $T \subseteq S$, it suffices to show that $\kappa'_T \leqslant \rho^a$ for any odd set T containing v^a. Let T be a set containing v^a with $|T|$ odd, $|T| \geqslant 3$ and $|S \setminus T| \geqslant 3$. By the same argument as in Lemma 1.7, we have

$$\begin{aligned} 2(w-x)(E(T)) &= 2\rho'(|T|-1) - \kappa'_S(|S|-1) + \kappa'_{S\setminus T}(|S|-|T|) \\ &= (2\rho' - \kappa'_S)(|T|-1) + (\kappa'_{S\setminus T} - \kappa'_S)(|S|-|T|). \end{aligned} \tag{1.11}$$

On the other hand, since $\mathcal{M}$ is a perfect matching, we have

$$\begin{aligned} |S\setminus T| = |S| - |T| + 1 &\leqslant 2|\mathcal{M} \cap E(S\setminus T)| + |T| \\ &= 2w(E(S\setminus T)) - 2(w-z)(E(S\setminus T)) + |T| \\ &= (\kappa_{S\setminus T} - \kappa'_{S\setminus T})(|S|-|T|) + |T|. \end{aligned}$$

By subtracting $(\kappa_{S\setminus T} - \kappa'_{S\setminus T})(|S|-|T|)+1$ from both sides of the last inequality, we obtain

$$(\kappa'_{S\setminus T} + 1 - \kappa_{S\setminus T})(|S|-|T|) \leqslant |T| - 1. \tag{1.12}$$

The restriction of $\mathcal{M}$ to $\mathcal{G}_S$ is a near perfect matching; therefore $\kappa'_S = \kappa_S - 1$. On the other hand, $\kappa_{S\setminus T} \leqslant \kappa_S$ since $\mathcal{G}_S$ is a κ-graph. These observations and (1.12) imply that

$$(\kappa'_{S\setminus T} - \kappa'_S)(|S|-|T|) \leqslant (\kappa'_{S\setminus T} - (\kappa_{S\setminus T} - 1))(|S|-|T|) \leqslant |T| - 1.$$

Substituting into (1.11), we obtain

$$\begin{aligned} 2(w-x)(E(T)) &\leqslant (2\rho' - \kappa'_S)(|T|-1) + (|T|-1) \\ &\leqslant (\rho'+1)(|T|-1) \qquad \text{since} \quad \rho' < \kappa'_S \\ &= \rho^a(|T|-1) \qquad \text{since} \quad \rho^a = \rho' + 1. \end{aligned}$$

We have thus shown that $\kappa'_T \leqslant \rho^a$ for every odd set T containing v^a and such that $|S\setminus T| \geqslant 3$. One would show in a similar fashion that $\kappa'_T \leqslant \rho^a$ for every odd set T containing v^a and such that $|S\setminus T| = 1$. This completes the proof of the lemma. ∎

Lemma 1.10 implies that if one is able to find a near perfect matching of $\mathscr{G}_S$ satisfying the condition of the lemma, then one is also able to construct a perfect matching of $\mathscr{G}_S^a$ such that the fractional chromatic index of $\mathscr{G}'$ is less than or equal to ρ^a. This observation and the observations preceding the statement of Lemma 1.10 lead us to formulate the following conjecture:

Conjecture 1.13. (A) Let $\mathscr{G}_S = (S, E, w)$ be a κ-graph. For every vertex v such that $d(v) < \lceil \kappa_S \rceil - 1$, there exists a near perfect matching $\mathscr{M}$ of $\mathscr{G}_S$ (whose incidence vector we shall denote by y) such that

(i) $\mathscr{M}$ is deficient at v;

(ii) $\dfrac{(w-y)(E(T))}{(|T|-1)/2} \leqslant \lceil \kappa_S \rceil - 1$ for every subset T of S with $|T|$ odd and $|T| \geqslant 3$.

(B) Let $\mathscr{G}$ be a κ-graph such that $\kappa_S \leqslant \rho + 1$ and $d(v) = \rho$ for every $v \in S$ (that is, $\mathscr{G}_S$ is a regular multigraph). Then the chromatic index of $\mathscr{G}_S$ is equal to $\rho + 1$.

We show that Conjecture 1.13 implies Seymour's conjecture. Contrary to what is the case for Conjecture 1.9, it is stronger than Seymour's conjecture. We think that Conjecture 1.13 is useful, however, since it pertains to critical graphs, on one hand, and will be used in Section 3, on the other.

THEOREM 1.14. *Let $\mathscr{G} = (V, E, w)$ be an arbitrary multigraph. If Conjecture 1.13 is true, the chromatic index of $\mathscr{G}$ is less than or equal to* $\max\{\rho + 1, \lceil \kappa \rceil\}$, *where ρ is the maximum degree of a vertex in $\mathscr{G}$ and*

$$\kappa = \max\left\{\frac{w(E(T))}{(|T|-1)/2} \,\middle|\, T \subseteq V,\ |T| \text{ odd and } |T| \geqslant 3\right\}.$$

Proof. The proof is identical to that of Theorem 1.8, except in the case where $\kappa > \rho$ and $|S| = |V|$ or $|V| - 1$. In this case, the proof of Theorem 1.8 uses Conjecture 1.9 to conclude that the chromatic index of $\mathscr{G}$ is equal to $\lceil \kappa \rceil$. Here we argue as follows: $\mathscr{G}$ can be embedded into the augmented κ-graph $\mathscr{G}_S^a = (S \cup \{v^a\}, E^a, w^a)$. If the degree of v^a in $\mathscr{G}_S^a$ is equal to zero, by definition of an augmented κ-graph, $\mathscr{G}_S^a$ is regular of degree ρ and such that $\kappa_S \leqslant \rho + 1$. By Conjecture 1.13(B), the chromatic index of $\mathscr{G}_S$ (and hence of $\mathscr{G}_S^a$) is equal to $\rho + 1 = \lceil \kappa \rceil$. This verifies Seymour's conjecture when $d(v^a) = 0$.

On the other hand, if the degree of v^a in $\mathscr{G}_S^a$ is positive, there exists some

$u \in S$ such that $w_e^a > 0$ for $e = \{v^a, u\}$. By Conjecture 1.13(A) there exists a near perfect matching $\mathcal{M}$ of $\mathcal{G}_S$ (whose incidence vector we denote by x) such that $\mathcal{M}$ is deficient at u and

$$\kappa'_T = \frac{(w^a - x)(E(T))}{(|T|-1)/2} \leqslant \lceil \kappa_S \rceil - 1 = \rho^a \qquad \text{for every odd subset } T \text{ of } S.$$

Let $\mathcal{G}'$ be the multigraph $(S \cup \{v^a\}, E^a, w^a - y)$, where y is the incidence vector of $\mathcal{M} \cup \{v^a, u\}$. Clearly $\rho' = \rho^a - 1$ and by Lemma 1.10, $\kappa' \leqslant \rho^a$. Therefore $\max\{\rho' + 1, \kappa'\} = \rho^a$, and by the induction hypothesis the chromatic index of $\mathcal{G}$ is given by $\max\{\rho' + 1, \lceil \kappa' \rceil\} + 1 = \max\{\rho + 1, \lceil \kappa \rceil\}$. We conclude that Seymour's conjecture is also verified when $d(v^a) > 0$. ∎

Thus Theorem 1.14 shows that the difference between the chromatic index of $\mathcal{G}$ and its fractional chromatic index depends upon the value of the chromatic index for certain critical multigraphs. Actually the second part of Conjecture 1.13 can be weakened as follows: let us assume that the chromatic index of $\mathcal{G}$ is $\rho + d$ for any multigraph $\mathcal{G}$ satisfying the hypotheses of the conjecture. Then an argument analogous to the argument of Theorem 1.14 shows that the difference between the chromatic index of $\mathcal{G}$ and its fractional chromatic index is less than or equal to d.

To conclude this section, we observe that in all likelihood, it will be difficult to prove either Seymour's conjecture or Conjecture 1.13. Nonetheless, the reduction of Seymour's conjecture to Conjecture 1.9 or Conjecture 1.13 remains valid if we restrict our attention to classes of graphs which can be defined in terms of excluded minors. This follows from the fact that the only "operations" used in the reduction are the shrinking operation (equivalent to the contraction of the edges of a vertex-induced subgraph) and the removal of a matching (equivalent to the delection of certain edges). These remarks can be summarized in the following theorem:

THEOREM 1.15. *Let $\mathscr{C}$ be a class of graphs which possesses an excluded minor characterization, and let us assume that the analogue of either Conjecture* 1.9 *or Conjecture* 1.13 *holds for graphs belonging to $\mathscr{C}$. Then Seymour's conjecture is verified for all graphs in $\mathscr{C}$.*

2. COMPATIBLE SUBGRAPHS

In Section 1, we showed that in order to prove that Seymour's conjecture is verified, it suffices to consider critical graphs. Our aim is to prove that the conjecture holds for outerplanar graphs (this class of graphs will be defined below). The difficulty in proving Conjecture 1.13(A) lies in the fact that κ-graphs may contain many subgraphs induced by subsets T of vertices such that $w(E(T)) > (\lceil \kappa_S \rceil - 1)((|T| - 1)/2)$. In this section we demonstrate that only certain subsets (hereafter called compatible subsets) need be considered.

DEFINITION 2.1. Let $\mathscr{G} = (S, E, w)$ be a multigraph such that $|S|$ is odd, and let T be a proper subset of S which induces a critical and nonseparable subgraph of $\mathscr{G}$. Let $\mathscr{G}^1$ be the graph obtained from $\mathscr{G}$ by shrinking T. We say that S and T are *compatible* if $\mathscr{G}^1$ is critical.

THEOREM 2.2. *Let $\mathscr{G}_S = (S, E, w)$ be a κ-graph, and v^0 be a vertex of $\mathscr{G}_S$ such that $d(v^0) < \lceil \kappa_S \rceil - 1$. Let $\mathscr{M}$ be a near perfect matching of $\mathscr{G}_S$ deficient at v^0, and x the incidence vector of $\mathscr{M}$. If $\mathscr{G}' = (S, E, w - x)$ is not a κ-graph, then there exists a subset T of S such that*

(i) *S and T are compatible;*

(ii) *$\mathscr{M} \cap E(T)$ is not a near perfect matching of the subgraph induced by T.*

Proof. Let $\kappa = \kappa_S$. By an argument similar to that of Lemma 1.4, one may show that w/κ belongs to the matching polyhedron. On the other hand, $(w - x)/(\kappa - 1)$ does not belong to the matching polyhedron. Let us now consider convex combinations of these two vectors. In order for the convex combination $\mu(w/\kappa) + (1 - \mu)((w - x)/(\kappa - 1))$ to belong to the matching polyhedron, we must have

$$\mu \geqslant \frac{(|T| - 1)/2 - (w - x)(E(T))/(\kappa - 1)}{w(E(T))/\kappa - (w - x)(E(T))/(\kappa - 1)}$$

for every critical nonseparable subset T such that

$$\frac{w(E(T))}{\kappa} - \frac{(w - x)(E(T))}{\kappa - 1} < 0.$$

Let us take

$$\mu = \max \left\{ \frac{(|T| - 1)/2 - (w - x)(E(T))/(\kappa - 1)}{w(E(T))/\kappa - (w - x)(E(T))/(\kappa - 1)} \,\middle|\, \frac{w(E(T))}{\kappa} - \frac{(w - x)(E(T))}{\kappa - 1} < 0 \right\}.$$

Then it is easily verified that

(i) $0 \leqslant \mu \leqslant 1$ (since $w(E(T))/\kappa \leqslant (|T| - 1)/2$ for every T and $(|T| - 1)/2 - (w - x)(E(T))/(\kappa - 1) < 0$ for at least one T);

(ii) $y = (w/\kappa) + (1 - \mu)((w - x)/(\kappa - 1))$ belongs to the matching polyhedron;

(iii) in particular $y(\delta(v)) < 1$ for every $v \in V$ because $\mathscr{G}$ is a κ-graph and $\mathscr{M}$ is deficient at a vertex v^0 such that $d(v^0) < \lceil \kappa_S \rceil - 1$, and

(iv) $y(E(S)) = (|S|-1)/2$ and $y(E(T)) = (|T|-1)/2$ for at least one proper subset of S which induces a critical nonseparable subgraph of $\mathscr{G}_S$. Let T denote such a subset.

It follows from (ii), (iii) and (iv) that S and T are compatible. For y can be written as $(1/p)(\sum_{i=1}^{p} x^i)$, where x^i is the incidence vector of a matching of $\mathscr{G}_S$ (see for instance Theorem 3.6 in Seymour [7]). (iv) implies that $x^i(E(S)) = (|S|-1)/2$ and $x^i(E(T)) = (|T|-1)/2$ for every i. (iii) implies that for each $v \in S$, there exists some i for which x^i is the incidence vector of a near perfect matching of $\mathscr{G}_S$ deficient at v. Therefore we conclude that for every vertex $v \in V$, there exists a matching $\mathscr{N}$ such that $\mathscr{N}$ is a near perfect matching of S deficient at v, and $\mathscr{N} \cap E(T)$ is a near perfect matching of the subgraph induced by T. It follows easily from this observation that the graph $\mathscr{G}^1$ obtained from $\mathscr{G}$ by shrinking T is critical; hence S and T are compatible.

Finally, the relation $w(E(T))/\kappa - (w-x)(E(T))/(\kappa - 1) < 0$ implies that $|\mathscr{M} \cap E(T)| < (|T|-1)/2$, i.e., $\mathscr{M} \cap E(T)$ is not a near perfect matching of the subgraph induced by T. ∎

We shall see below that the family of compatible subsets of an outerplanar graph has a very special structure, i.e., it is essentially a nested family of critical subsets (see Pulleyblank [6]). In the next section we shall need the following lemmas.

LEMMA 2.3. *Let $\mathscr{G}_S = (S, E, w)$ be a κ-graph, and let T be a proper subset of S such that*

(i) *the subgraph induced by T is critical and nonseparable*;

(ii) *$\kappa_T > \rho$ and $\kappa_T \geqslant \kappa_W$ for any odd subset W such that $T \subset W \subset S$.*

Then $\mathscr{G}^1$, the graph obtained from $\mathscr{G}$ by shrinking T, is a κ-graph.

Proof. Let $\mathscr{G}^1 = (V^1, E^1, w^1)$ be the graph obtained from $\mathscr{G}$ by shrinking T. We can show, by the same argument as that given in Lemma 1.7, that the degree of v_P in $\mathscr{G}^1$ is less than ρ, the maximum degree of $\mathscr{G}$. On the other hand, one can show easily that

(1) if W is an odd subset of V^1 not containing v_P, then $w^1(E^1(W))/((|W|-1)/2) = \kappa_W$;

(2) if W is an odd subset of S such that $T \subset W$, then $w^1(E^1(Z))/((|Z|-1)/2) \leqslant \kappa_W$, where $Z = (W \setminus T) \cup \{v_P\}$;

(3) $w^1(E^1)/((|V^1|-1)/2) \geqslant \kappa_S$.

We conclude that $\mathscr{G}^1$ is a κ-graph. ∎

LEMMA 2.4. *Let $\mathscr{G}_S = (S, E, w)$ be a κ-graph, and let T be a proper subset of S such that*

(i) *the subgraph induced by T is critical and nonseparable;*

(ii) $\kappa_T > \rho$.

Then S and T are compatible.

Proof. We show that S and T are compatible by induction on $|S \setminus T|$. If $|S \setminus T| = 2$, $\mathscr{G}^1$ is obviously a κ-graph, since $\mathscr{G}^1$ does not have any proper odd subset of cardinality greater than 1. If $|S \setminus T| > 2$ and $\mathscr{G}^1$ is a κ-graph, $\mathscr{G}^1$ is critical by Theorem 1.2, and we conclude that S and T are compatible. If $|S \setminus T| > 2$ and $\mathscr{G}^1$ is not a κ-graph, there exists a proper subset W of V^1 such that v_P belongs to W and the subgraph of $\mathscr{G}^1$ induced by W is a κ-graph. Let X be the set $(W \setminus \{v_P\}) \cup T$. It is easy to verify that $\mathscr{G}_X$ is critical and nonseparable and that κ_T is greater than ρ. Since $|S \setminus X|$ is smaller than $|S \setminus T|$, we may apply the induction hypothesis to conclude that S and X are compatible. Finally, X and T are compatible because $\mathscr{G}^1$ is critical. Therefore S and T are compatible in this case also. ∎

We now turn to consideration of outerplanar multigraphs. Let $\mathscr{G} = (V, E, w)$ be a multigraph. We say that $\mathscr{G}$ is outerplanar if it can be embedded in the plane in such a fashion that all its vertices lie on the exterior face (see Chartrand and Harary [2]). The following observations are easily verified:

(A) An outerplanar graph is nonseparable if and only if it is hamiltonian. In particular, an outerplanar κ-graph is hamiltonian.

(B) Let $\mathscr{G}$ be an outerplanar graph with V a set of odd cardinality. Let us assume that there exists a vertex v such that v belongs to every

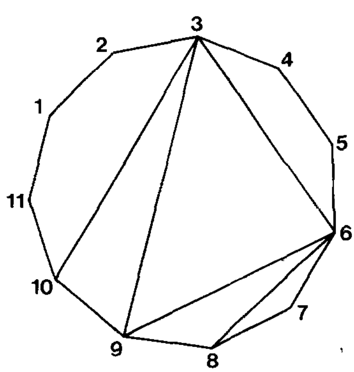

FIGURE 2.5

biconnected component of $\mathcal{G}$. Then $\mathcal{G}$ is critical if and only if all of its biconnected components have odd cardinality.

(C) Let $\mathcal{G}$ be an outerplanar multigraph, and T an odd subset of vertices which induces a critical nonseparable subgraph of $\mathcal{G}$. Then both the subgraph induced by T and the graph obtained from $\mathcal{G}$ by "shrinking" T are outerplanar graphs (this is an easy consequence of the excluded minor characterization of outerplanar graphs given by Chartrand and Harary [2]).

It follows from observations (B) and (C) that if $\mathcal{G}_S$ is an outerplanar graph and T is an odd set inducing a critical nonseparable subgraph of $\mathcal{G}_S$, then S and T are compatible if and only if every biconnected component of $\mathcal{H}$ has an odd number of vertices (where $\mathcal{H}$ denotes the graph obtained from $\mathcal{G}$ by shrinking T). For instance, let S denote the vertex set of the graph of Fig. 2.5, and let T be $\{3, 6, 9\}$ and W be $\{3, 6, 8, 9, 10\}$. S and T are compatible, while S and W are not.

The following theorem will be used in the next section to prove that Seymour's conjecture is satisfied in the case of outerplanar graphs.

THEOREM 2.6. *Let $\mathcal{G} = (S, E, w)$ be an outerplanar nonseparable multigraph with an odd number of vertices. For every vertex v of $\mathcal{G}$, there exists a near perfect matching $\mathcal{M}$ of $\mathcal{G}$ deficient at v such that*

$$|\mathcal{M} \cap E(T)| = \frac{|T| - 1}{2}$$

for every odd set T such that S and T are compatible.

Proof. It follows from the above remarks that $\mathcal{G}$ is hamiltonian. If E is the edge set of an odd cycle, there is no proper subset of S which induces a critical subgraph of $\mathcal{G}$. Thus the theorem is verified in that case. If E is not the edge set of an odd cycle, then there exists a set $W = \{v_1, v_2, \ldots, v_r\}$ such that

(i) W is a proper subset of S;

(ii) $E(W)$ is the edge set of a cycle, that is, $E(W)$ consists of edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_r, v_1\}$;

(iii) all the edges of $E(W)$ except one belong to the hamiltonian cycle of $\mathcal{G}$. We shall assume that the edge $\{v_1, v_2\}$ does not belong to the hamiltonian cycle of $\mathcal{G}$.

It should be clear that for any subset T of S such that $E(W) \cap E(T) \neq \emptyset$ and the subgraph induced by T is nonseparable, we have either $T \cap W = \{v_1, v_2\}$ or $T \supseteq W$.

In order to construct a near perfect matching verifying the conclusion of the theorem, we consider two cases:

1. *W Has Odd Cardinality*

In this case $E(W)$ is the edge set of an odd cycle, and the subgraph induced by W is critical. Let $\mathscr{G}^1$ be the graph obtained from $\mathscr{G}$ by shrinking W. $\mathscr{G}^1$ is an outerplanar nonseparable graph with an odd number of vertices. By induction we may assume that for any vertex v of $\mathscr{G}^1$, there exists a near perfect matching $\mathscr{M}^1$ of $\mathscr{G}^1$ deficient at v and such that

$$|\mathscr{M}^1 \cap E(T)| = \frac{|T|-1}{2} \qquad \text{for every odd set } T \text{ such that } V^1 \text{ (the vertex set of } \mathscr{G}^1\text{) and } T \text{ are compatible.} \tag{2.7}$$

Let v be any vertex of $\mathscr{G}$. If $v \in W$, let $\mathscr{M} = \mathscr{M}^1 \cup \mathscr{M}^2$, where $\mathscr{M}^2$ is the near perfect matching of W deficient at v, and $\mathscr{M}^1$ is a near perfect matching of $\mathscr{G}^1$ satisfying 2.7 and deficient at v_P, the pseudo-vertex of $\mathscr{G}^1$ which corresponds to W. If $v \notin W$, let $\mathscr{M} = \mathscr{M}^1 \cup \mathscr{M}^2$, where $\mathscr{M}^1$ is a near perfect matching of $\mathscr{G}^1$ satisfying 2.7 and deficient at v, and $\mathscr{M}^2$ is the near perfect matching of W deficient at v^0 (v^0 is the endpoint in W of the edge of $\mathscr{M}^1$ which contains v_P). We claim that for any T such that S and T are compatible, $\mathscr{M} \cap E(T) = (|T|-1)/2$.

(a) $E(T) \cap E(W) = \varnothing$. Then the subgraph induced by T is a subgraph of $\mathscr{G}^1$, and we have $|\mathscr{M} \cap E(T)| = |\mathscr{M}^1 \cap E(T)| = (|T|-1)/2$ since $\mathscr{M}^1$ satisfies (2.7).

(b) $E(T) \cap E(W) \neq \varnothing$. If $T \cap W = \{v_1, v_2\}$, S and T are not compatible, since $\{v_3, \ldots, v_r, v_P\}$ is a biconnected component of $\mathscr{G}^2$ which has even cardinality (where $\mathscr{G}^2$ denotes the graph obtained from $\mathscr{G}$ by shrinking T). Thus the only possibility is $T \supseteq W$. Let T^1 be the set $((T \setminus W) \cup \{v_P\}$. Then V^1 and T^1 are compatible, and

$$\begin{aligned} |\mathscr{M} \cap E(T)| &= |\mathscr{M}^2| + |\mathscr{M}^1 \cap E(T^1)| \\ &= \frac{|W|-1}{2} + \frac{|T^1|-1}{2} = \frac{|T|-1}{2}. \end{aligned}$$

2. *W Has Even Cardinality*

Let us consider the graph $\mathscr{H}$ induced by the set $V^2 = S \setminus \{v_3, \ldots, v_r\}$. $\mathscr{H}$ is an outerplanar nonseparable multigraph with an odd number of vertices. By the induction hypothesis we may assume that for any $v \in V^2$, there exists a near perfect matching $\mathscr{M}^2$ of $\mathscr{H}$ such that $\mathscr{M}^2$ is deficient at v and $|\mathscr{M}^2 \cap E(T)| = (|T|-1)/2$ for any subset T of V^2 such that V^2 and T are compatible. On the other hand, let $\mathscr{G}^1$ be the graph obtained from $\mathscr{G}$ by

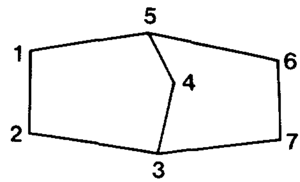

FIGURE 2.8

shrinking V^2. The edge set of $\mathscr{G}^1$ is an odd cycle, and for any vertex v^0 of $\mathscr{G}^1$, there is a near perfect matching $\mathscr{M}^1$ of $\mathscr{G}^1$ deficient at v^0. As in case 1, we construct a near perfect matching $\mathscr{M}$ deficient at $v \in V^2$ by taking the union of a matching $\mathscr{M}^2$ deficient at v and a matching $\mathscr{M}^1$ deficient at v_P; and we construct a near perfect mathing $\mathscr{M}$ deficient at $v \notin V^2$ by taking the union of a matching $\mathscr{M}^1$ deficient at v and a matching $\mathscr{M}^2$ deficient at v^0 (where v^0 is the endpoint in V^2 of the edge of $\mathscr{M}^1$ which is incident upon v_P). We now show that $\mathscr{M}$ verifies the conclusion of the theorem. Let T be a subset of S such that S and T are compatible. If T is a subset of V^2, then

$$|\mathscr{M} \cap E(T)| = |\mathscr{M}^2 \cap E(T)| = \frac{|T|-1}{2}.$$

If T is not a subset of V^2, T must contain W, since T is nonseparable. We have

$$\begin{aligned} |\mathscr{M} \cap E(T)| &= |\mathscr{M}^1| + |\mathscr{M}^2 \cap E(T \cap V^2)| \\ &= \frac{|T \setminus V^2|}{2} + \frac{|T \cap V^2| - 1}{2} = \frac{|T|-1}{2}, \end{aligned}$$

since V^2 and $T \cap V^2$ are compatible. This completes the proof of the theorem. ∎

It should be pointed out that the class of graphs for which a statement similar to that of Theorem 2.6 holds is rather small, since the graph of Fig. 2.8, for instance, does not belong to it. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$, $T = \{1, 2, 3, 4, 5\}$, and $W = \{3, 4, 5, 6, 7\}$. It is clear that there does not exist a near perfect matching $\mathscr{M}$ deficient at node 4 such that each of the subgraphs induced by the sets T and W contains exactly two edges of $\mathscr{M}$.

COROLLARY 2.9. *Let $\mathscr{G} = (S, E, w)$ be an outerplanar nonseparable multigraph, and let T and W be subsets of S such that S and T, on one hand, and S and W, on the other, are compatible. Then the following hold:*

(i) *if* $|T \cap W| > 0$, $T \cap W$ *has odd cardinality;*

(ii) *if* $|T \cap W| > 1$, S *and* $T \cap W$ *are compatible;*

(iii) *if* $|T \cap W| > 1$, T *and* $T \cap W$ *on one hand, and* W *and* $T \cap W$ *on the other, are compatible;*

(iv) *if* $|T \cap W| > 1$, S *and* $T \cup W$ *are compatible;*

(v) *let* $\mathscr{G}^1 = (V^1, E^1, w^1)$ *be the graph obtained from* $\mathscr{G}$ *by shrinking* W. *If* $T \cap W = \varnothing$, V^1 *and* T *are compatible. If* $T \cap W \neq \varnothing$, V^1 *and* $(T \setminus W) \cup \{v_P\}$ *are compatible.*

Proof. We shall prove only (i), since (ii), (iii), (iv), and (v) are proved in a similar fashion. Let us assume that $|T \cap W|$ is even and different from 0. Let v^0 be any vertex of $T \cap W$. By Theorem 2.6 there exists a near perfect matching of $\mathscr{G}$ deficient at v^0 such that $|\mathscr{M} \cap E(T)| = (|T| - 1)/2$ and $|\mathscr{M} \cap E(W)| = (|W| - 1)/2$. But this contradicts Proposition 5.1.7 of Pulleyblank [6], since the latter implies that any matching such that

$$|\mathscr{M} \cap E(T)| = \frac{|T| - 1}{2} \qquad \text{and} \qquad |\mathscr{M} \cap E(W)| = \frac{|W| - 1}{2}$$

must contain an edge incident upon v for every vertex v belonging to $T \cap W$. ∎

3. A Proof of Seymour's Conjecture for the Case of Outerplanar Graphs

We are now ready to tackle the proof of Seymour's conjecture for outerplanar graphs. It follows easily from Chartrand and Harary [2] that a graph is outerplanar if and only if it contains no K_4 minor and no $K_{2,3}$ minor. Therefore, by Theorem 1.15, it suffices to prove that Conjecture 1.13 holds for outerplanar graphs in order to verify Seymour's conjecture. The first part of Conjecture 1.13 for outerplanar graphs is an easy consequence of Theorems 2.2 and 2.6. Actually, Theorems 2.2 and 2.6 imply the following theorem, which is stronger than the first part of Conjecture 1.13.

THEOREM 3.1. *Let* $\mathscr{G}_S = (S, E, w)$ *be a* κ*-graph which is also outerplanar. For every vertex* v *such that* $d(v) < \lceil \kappa_S \rceil - 1$, *there exists a near perfect matching* $\mathscr{M}$ *of* $\mathscr{G}_S$ *(whose incidence vector we shall denote by* y*) such that*

(i) $\mathscr{M}$ *is deficient at* v;

(ii) $\dfrac{(w-y)(E(T))}{((|T|-1)/2)} \leqslant \kappa_S - 1$ *for every subset* T *of* S *such that* $|T|$ *is odd and* $|T| \geqslant 3$.

Hence the graph $\mathscr{G}' = (S, E, w - y)$ *is a* κ*-graph.*

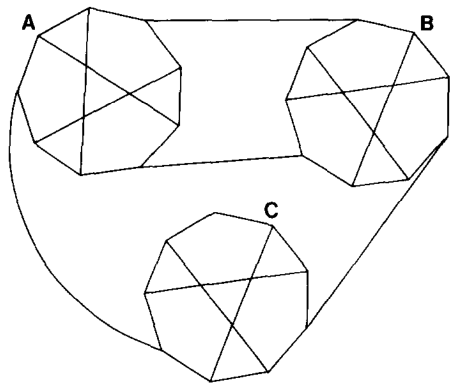

FIGURE 3.2

Proof. By Theorem 2.6 there exists a near perfect matching $\mathscr{M}$ of $\mathscr{G}_S$ which is deficient at v and such that $|\mathscr{M} \cap E(T)| = (|T| - 1)/2$ for every set T such that S and T are compatible. Let us assume that the conclusion of the theorem fails. Then the graph $\mathscr{G}' = (S, E, w - y)$, where y is the incidence vector of $\mathscr{M}$, is not a κ-graph. By Theorem 2.2 there exists a set W such that S and W are compatible and $\mathscr{M} \cap E(W)$ is not a near perfect matching of the subgraph induced by W. But this contradicts the choice of $\mathscr{M}$. Therefore the conclusion of the theorem holds. ∎

Theorem 3.1 clearly implies the first part of Conjecture 1.13. The conclusion of the theorem, however, does not hold in general. Consider the graph of Fig. 3.2. Each of the subgraphs (A), (B), and (C) is the Petersen graph from which one vertex has been removed. It is not possible to remove a near perfect matching from the graph of Fig. 3.2 in such a way that the resulting graph is a κ-graph. Thus the conclusion of Theorem 3.1 (but not the first part of Conjecture 1.13) fails for this graph.

We shall now prove that the second part of Conjecture 1.13 holds as well for outerplanar graphs. The gist of the proof is to remove some edges from a regular graph in order to apply Theorem 3.1 once more. In order to do this we need the following (technical) lemma.

LEMMA 3.3. *Let $\mathscr{G}_S = (S, E, w)$ be an outerplanar κ-graph with $\kappa_S \leqslant \rho + 1$ (where ρ denotes the maximum degree of $\mathscr{G}$). Let us assume that*

$$w(E) - w(E(T)) > (\rho + 1)\left(\frac{|S \setminus T|}{2}\right) \qquad \text{for every subset } T \text{ of } S \text{ such that } \kappa_T > \rho. \tag{3.4}$$

Then there exists a matching $\mathcal{M}$ of G satisfying

(i) $|\mathcal{M}| = w(E) - \rho((|S|-1)/2)$;

(ii) $|\mathcal{M} \cap E(T)| \geqslant w(E(T)) - \rho((|T|-1)/2) + 1$ *for every* T *with* $\kappa_T > \rho$;

(iii) $\mathcal{M}$ *contains at least one edge belonging to the hamiltonian cycle of* $\mathcal{G}$.

Proof. If there does not exist any proper subset T of S such that $\kappa_T > \rho$, let $\mathcal{M}^1$ be any near perfect matching of $\mathcal{G}$. $\mathcal{M}^1$ contains at least one edge belonging to the hamiltonian cycle of $\mathcal{G}$. The hypothesis that $\kappa_S \leqslant \rho + 1$ implies that $w(E) - \rho((|S|-1)/2) \leqslant (|S|-1)/2$. We may thus take $\mathcal{M}$ to be any subset of $\mathcal{M}^1$ of cardinality $w(E) - \rho((|S|-1)/2)$ which contains at least one edge belonging to the hamiltonian cycle of $\mathcal{G}$. It is clear that $\mathcal{M}$ verifies the conclusion of the theorem.

On the other hand, if there exists at least one proper subset T of S with $\kappa_T > \rho$, we choose a proper subset W of S such that

(a) $\kappa_W \geqslant \kappa_T$ for every proper subset T of S of odd cardinality;

(b) W is minimal with respect to property (a).

It is clear that the subgraph induced by W is a κ-graph, hence a non-separable subgraph of $\mathcal{G}$. By Lemma 2.3, $\mathcal{G}^1$, the graph obtained by shrinking W, is a κ-graph. By Theorem 2.6, there exists a near perfect matching $\mathcal{M}^1$ of $\mathcal{G}^1$ such that $|\mathcal{M}^1 \cap E^1(T)| = (|T|-1)/2$ for every subset T of V^1 such that V^1 and T are compatible. Furthermore, $\mathcal{M}^1$ contains at least one edge belonging to the hamiltonian cycle of $\mathcal{G}^1$, since $\mathcal{M}^1$ is a near perfect matching of $\mathcal{G}^1$. Since the edges of the hamiltonian cycle of $\mathcal{G}^1$ belong to the hamiltonian cycle of $\mathcal{G}$, $\mathcal{M}^1$ contains at least one edge belonging to the hamiltonian cycle of $\mathcal{G}$.

We now consider the subgraph of $\mathcal{G}$ induced by the set W. Let $t = (\kappa - \rho)((|S|-1)/2) - |S \setminus W|/2$. We have

$$\kappa\left(\frac{|S|-1}{2}\right) - \kappa_W\left(\frac{|W|-1}{2}\right) > (\rho+1)\left(\frac{|S \setminus W|}{2}\right) \qquad \text{by (3.4)}$$

which implies that

$$(\kappa-\rho)\left(\frac{|S|-1}{2}\right) > \frac{|S \setminus W|}{2} + (\kappa_W - \rho)\left(\frac{|W|-1}{2}\right).$$

Thus t is a positive integer. We also have $\kappa \leqslant \rho + 1$, from which

$$(\kappa-\rho)\left(\frac{|S|-1}{2}\right) \leqslant \frac{|S|-1}{2}$$

and

$$(\kappa-\rho)\left(\frac{|S|-1}{2}\right)-\frac{|S\setminus W|}{2}\leqslant\frac{|W|-1}{2}$$

follow. Hence t is less than or equal to the cardinality of a near perfect matching of W. Let $\mathcal{M}^2$ be a near perfect matching of W such that $|\mathcal{M}^2\cap E(T)|=(|T|-1)/2$ for every subset T of W such that W and T are compatible, and let $\mathcal{M}^3$ be any subset of $\mathcal{M}^2$ which has cardinality t (such a subset may always be found, since $0<t\leqslant(|W|-1)/2$).

We claim that $\mathcal{M}=\mathcal{M}^1\cup\mathcal{M}^3$ verifies the conclusion of the theorem. Lemma 3.3(i) is clearly satisfied, since

$$\mathcal{M}=\frac{|S\setminus W|}{2}+t=(\kappa-\rho)\left(\frac{|S|-1}{2}\right)=w(E)-\rho\left(\frac{|S|-1}{2}\right).$$

Lemma 3.3(iii) is also satisfied, since $\mathcal{M}^1$ contains an edge belonging to the hamiltonian cycle of $\mathcal{G}$. Let T be a subset of S which induces a critical non-separable subgraph of $\mathcal{G}$, and let us assume that $\kappa_T>\rho$. By Lemma 2.4, S and T are compatible. If $E(T)\cap E(W)=\varnothing$ we have

$$\begin{aligned}|\mathcal{M}\cap E(T)|&=|\mathcal{M}^1\cap E(T)|\\&=\frac{|T|-1}{2}\qquad\text{since } V^1 \text{ and } T \text{ are compatible by Corollary 2.9}\\&\geqslant w(E(T))-\rho\left(\frac{|T|-1}{2}\right)+1.\end{aligned}$$

The last inequality follows from the fact that $\kappa_T<\rho+1$, since $\mathcal{G}$ is a κ-graph. On the other hand, let T be such that $E(T)\cap E(W)\neq\varnothing$. We assume that $T\setminus W\neq\varnothing$ since the case $T\subseteq W$ can be treated in a similar fashion. Then $|T\cap W|\geqslant 2$, and by Corollary 2.9, W and $T\cap W$ are compatible. We have

$$\begin{aligned}|\mathcal{M}\cap E(T)|&=|\mathcal{M}^1\cap E(Z)|+|\mathcal{M}^3\cap E(T\cap W)|\\&\qquad\text{where } Z=(T\setminus W)\cup\{v_P\}\\&=\frac{|T\setminus W|}{2}+|\mathcal{M}^3\cap E(T\cap W)|\\&\qquad\text{since } V^1 \text{ and } Z \text{ are compatible by Corollary 2.9}\\&\geqslant\frac{|T\setminus W|}{2}+\max\left\{t-\frac{|W\setminus T|}{2},0\right\}\end{aligned}$$

since $\mathcal{M}^3$ is a subset of $\mathcal{M}^2$
and $\mathcal{M}^2 \cap E(T \cap W) = (|T \cap W| - 1)/2$

$$\geqslant (\kappa - \rho)\left(\frac{|S|-1}{2}\right) + \left(\frac{|T \setminus W|}{2} - \frac{|S \setminus W|}{2} - \frac{|W \setminus T|}{2}\right)$$

$$> (\kappa_T - \rho)\left(\frac{|T|-1}{2}\right) + \frac{|S \setminus T|}{2} + \frac{|T \setminus W|}{2} - \frac{|S \setminus W|}{2} - \frac{|W \setminus T|}{2}$$

by (3.4)

$$= w(E(T)) - \rho\left(\frac{|T|-1}{2}\right).$$

Since $|\mathcal{M} \cap E(T)|$ is an integer

$$|\mathcal{M} \cap E(T) \geqslant w(E(T)) - \rho\left(\frac{|T|-1}{2}\right) + 1.$$

This completes the proof of the lemma. ▮

We are now ready to prove that the second part of Conjecture 1.13 is verified as well in the case of outerplanar graphs.

THEOREM 3.5. *Let $\mathcal{G}_S = (S, E, w)$ be an outerplanar κ-graph such that $\kappa_S \leqslant \rho + 1$ and $d(v) = \rho$ for every $v \in S$. Then the chromatic index of $\mathcal{G}_S$ is equal to $\lceil \kappa_S \rceil = \rho + 1$.*

Proof. We assume that Seymour's conjecture is true for all graphs whose vertex set has cardinality less than $|S|$ (the conjecture is trivially true for graphs with three vertices). There are two cases to consider:

(1) There exists a proper subset T of S such that $\kappa_T > \rho$ and

$$w(E) - w(E(T)) \leqslant (\rho + 1)\left(\frac{|S \setminus T|}{2}\right).$$

Let W be a proper subset of S such that

(a) $\kappa_W > \rho$;

(b) $w(E) - w(E(W)) \leqslant (\rho + 1)(|S \setminus W|/2)$;

(c) W is maximal with respect to (a) and (b).

It is clear that $\kappa_T \leqslant \rho + 1$ for every subset T of W which induces a critical nonseparable subgraph of W. By the induction hypothesis the chromatic index of this subgraph is $\rho + 1$. Let us now consider the graph $\mathcal{G}^1 = (V^1, E^1, w^1)$ obtained from $\mathcal{G}_S$ by shrinking W. It easily follows from

(b) and (c) that $w(E(Z))-w(E(W))\leqslant(\rho+1)(|Z\setminus W|/2)$ for every set Z which contains W and induces a critical nonseparable graph. This in turn implies that $\kappa_T^1=w^1(E(T))/((|T|-1)/2)$ is less than or equal to $\rho+1$ for every odd subset T of V^1. We may thus apply the induction hypothesis again, and we conclude that the chromatic index of $\mathscr{G}^1$ is $\rho+1$. By an argument similar to that given in Lemma 1.6, the colorings of $\mathscr{G}^1$ and of the subgraph induced by W may be combined to produce a proper coloring of $\mathscr{G}$ containing $\rho+1$ colors.

(2) There does not exist any proper subset T of S such that $\kappa_T>\rho$ and $w(E)-w(E(T))\leqslant(\rho+1)(|S\setminus T|/2)$.

In this case the hypotheses of Lemma 3.3 are satisfied, and we conclude that there exists a matching $\mathscr{M}$ of $\mathscr{G}$ satisfying

(i) $|\mathscr{M}|=w(E)-\rho((|S|-1)/2)$;

(ii) $|\mathscr{M}\cap E(T)|\geqslant w(E(T))-\rho((|T|-1)/2)+1$ for every T with $\kappa_T>\rho$; and

(iii) $\mathscr{M}$ contains at least one edge belonging to the hamiltonian cycle of $\mathscr{G}$; let us denote one of these edges by e.

Let $\mathscr{M}^0$ be $\mathscr{M}\setminus\{e\}$, x the incidence vector of $\mathscr{M}^0$ and $\mathscr{G}'$ the graph $(S, E, w-x)$. Clearly the maximal degree of $\mathscr{G}'$ is ρ. We also have

$$(w-x)(E)=w(E)-|\mathscr{M}|+1>\rho\left(\frac{|S|-1}{2}\right),$$

$$(w-x)(E(T))\leqslant w(E(T))-|\mathscr{M}\cap E(T)|+1\leqslant\rho\left(\frac{|T|-1}{2}\right)$$

for every T such that $\kappa_T>\rho$

and

$$(w-x)(E(T))\leqslant w(E(T))\leqslant\rho\left(\frac{|T|-1}{2}\right)$$

for every T such that $\kappa_T\leqslant\rho$.

If $\mathscr{M}^0=\varnothing$, $\mathscr{G}'=\mathscr{G}_S$ is a κ-graph of degree 2, that is, a cycle, and the theorem holds. If $\mathscr{M}^0\neq\varnothing$, $\mathscr{G}'$ is a κ-graph which is not regular, and we may apply Theorem 3.1 in order to construct a family of matchings which partition the edge set of $\mathscr{G}'$. Let $\{v_1, v_2,..., v_l\}$ be the set of vertices which are endpoints of edges in $\mathscr{M}^0$. By repeated application of Theorem 3.1, we conclude that

$$w-x=\sum_{j=1}^{l}y^j+z,$$

where y^j is the incidence vector of a near perfect matching $\mathcal{M}^j$ deficient at v_j and z is the incidence vector of the hamiltonian cycle of $\mathcal{G}_S$. Thus $z_e = 1$. Let $\mathcal{N}^1$ and $\mathcal{N}^2$ be the two near perfect matchings such that $\mathcal{N}^1 \cup \mathcal{N}^2 \cup \{e\}$ is the hamiltonian cycle of $\mathcal{G}_S$. Then the collection $\{\mathcal{M}, \mathcal{N}^1, \mathcal{N}^2, \mathcal{M}^1, ..., \mathcal{M}^l\}$ is a proper coloring of $\mathcal{G}_S$. This shows that the chromatic index of $\mathcal{G}_S$ is $\rho + 1$, since $l + 2 = 2\,|\mathcal{M}| = \rho$. ∎

4. Conclusion

We have proved that Seymour's conjecture holds for outerplanar graphs by reducing this conjecture to a conjecture about certain critical non-separable graphs. The technique described in this paper might be useful for proving Seymour's conjecture in more general cases. In particular, since the class of outerplanar graphs is the class of graphs which do not have a K_4 minor or a $K_{2,3}$ minor, a proof that Seymour's conjecture holds for K_4-free graphs or for $K_{2,3}$-free graphs would be a natural extension of the result presented in this paper.

Acknowledgments

Part of the work presented in this paper was conducted while the author was a visiting scientist at the Thomas J. Watson Research Center in Yorktown Heights. She gratefully acknowledges the support given to her by the IBM Corporation and the University of Sherbrooke on this occasion.

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